Generation of maximally entangled states with sub-luminal Lorentz boosts

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Outline

Introduction

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Motivation

- Entanglement as resource in quantum information theory. Enables to create technology that performs information processing tasks which are beyond the limits of the classical realm.
- The theory of entanglement given in non-relativistic QM, i.e. Galilean spacetime. But spacetime is ultimately relativistic theory, hence need to study relativistic entanglement.

Historical background

- In 1926, Thomas discovers Thomas precession and derives the relativistic correction for the hydrogen atom [5, 6].
- In 1939, Wigner provides a comprehensive treatment of unitary representations for the Lorentz group [7].
- First papers on relativistic quantum information on single and two particle entanglement around 2000 [1, 4, 2, 3].
Entanglement

- The term entanglement (*Verschränkung*) was coined by Schrödinger (1935) in the discussion of the EPR (1935) experiment.
- Bell (1964) shows that entangled quantum states display correlations that cannot be reproduced by classical physics.
- Entanglement is regarded as the feature that distinguishes quantum and classical physics.
- Quantum information theory uses entangled states to perform information processing tasks that are beyond classical realm.
Entanglement II

* A quantum state $|\psi\rangle$ is entangled if it cannot be written as a product of its constituents. For bipartite systems

$$|\psi^{AB}\rangle \neq |\psi^A\rangle \otimes |\psi^B\rangle \equiv |\psi^A\rangle|\psi^B\rangle$$

* Example: for a two-level bipartite system, $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$
  - $|0\rangle|0\rangle, |0\rangle|1\rangle, \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle)$ etc are product
  - $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle)$ etc are entangled

* In general, characterization of entanglement is a non-trivial task. Bipartite entanglement is now well understood. The theory of multi-partite entanglement an active field of research
Galilean spacetime

- Galilei boosts form a group: for any two (non-collinear) Galilei boosts $G(v_1), G(v_2)$, there exists a third $G(v_3)$ boost such that

$$G(v_2)G(v_1) = G(v_3), \quad v_3 = v_1 + v_2.$$  \hspace{1cm} (2)

Adding velocities non-relativistically, $v_1 + v_2 = v_3$

Figure 1. Three Galilei frames.
Minkowski spacetime

- Lorentz boosts do not form a group: two non-collinear Lorentz boosts

\[ \Lambda(v_2)\Lambda(v_1) \neq \Lambda(v_3), \quad \Lambda(v_2)\Lambda(v_1) = R(\omega)\Lambda(v_3) \quad (3) \]

where \( R(\omega) \in SO(3) \) is Wigner’s rotation for massive particles

Rotation angle

\[ \tan \left( \frac{\omega}{2} \right) = \frac{\sin \theta}{\cos \theta + D(v_1, v_2)} \]

Kinematic effect due to spacetime structure, depends solely on the velocity and acceleration of the object

\[ \hat{n} = \hat{v}_2 \times \hat{v}_1 \]
Non-relativistic spacetime

\[ |\psi'\rangle_C = |\uparrow\rangle |p\rangle \]

\[ |\psi''\rangle_C = |\uparrow\rangle |p''\rangle \]

Figure 2. Non-relativistic frames.
Relativistic spacetime

\[ |\psi'\rangle_C = |\uparrow\rangle |p\rangle \]

\[ |\psi''\rangle_C = |\swarrow\rangle |p''\rangle \]

Figure 3. Relativistic frames.
Wigner rotation: example 1

- Bob and Alice to meet. Alice gives Bob instructions: $B_y(v_1)B_x(v_0)$. But Bob has a lot on his mind and takes instead the route $B_x(v_0)B_y(v_1)$. Alice and Bob never meet. . .

Figure 4. Two Lorentz boosts non-commutative.
Wigner rotation properties

Figure 5. Dependence of Wigner rotation on the angle \( \theta \) between two boosts.
Lorentz boosted single spin-1/2 particle

Focus on a single massive spin-1/2 particle with momentum
“Assuming that spin and momentum are initially in a product state, will they become entangled after two non-collinear Lorentz boosts?”

Generic state

\[ |\psi\rangle = \sum_{\lambda} \int \psi_{\lambda}(p)|p\rangle|\lambda\rangle \, d\mu(p) \]  

To Lorentz boosted observer \( O'' \) the state of the particle appears transformed

\[ \psi_{\lambda}(p) \mapsto \psi'_{\lambda}(p) = \sum_{\kappa} U_{\lambda\kappa}(R(\Lambda, \Lambda^{-1}p))\psi_{\kappa}(\Lambda^{-1}p) \]  

Example

\[ (|p\rangle + |-p\rangle) |0\rangle \xrightarrow{\Lambda} |p''\rangle |\swarrow\rangle + |-p''\rangle |\nearrow\rangle \]
Lorentz boosted single spin-1/2 particle

Since we are interested in spin, trace out the momentum

\[ \rho_S'' = \text{Tr}_p \left( U(\Lambda) \left| \psi \right\rangle \left\langle \psi \right| U^\dagger(\Lambda) \right) \]

\[ = \sum_{\lambda\kappa} \int \psi''(p) \psi''^*(p) |\lambda\rangle \langle \kappa| \, d\mu(p) \]  

(7)

To quantify entanglement, we calculate the von Neumann entropy of spin

\[ S(\rho''_S) = -\text{Tr}(\rho''_S \log \rho''_S) \]  

(8)
Lorentz boosted single spin-1/2 particle

- Rest frame state: wavefunction $x$-symmetric Gaussian centred at $p_0 = (\pm p_{x0}, 0, p_{z0})$, spin $z$-up and boost in the $z$-direction

Figure 6. Single spin-1/2 particle with spin and momentum. Lorentz boost in the $z$-direction.
Particle in different boost scenarios

Figure 7. (a) Spin entropy for an $x$-symmetric Gaussian with $\sigma/m = 1$ with boost geometry $\theta = 90^\circ$ and $v_1 = 0.985$. (b) Schematic representation of Gaussian in the rest frame.
Particle in different boost scenarios

Figure 8. (a) Spin entropy for $x$-symmetric Gaussians with $\sigma/m = 1$. Three boost geometries with different $\theta_i$ are shown, all $v_1 = 0.985$. (b) Schematic representation of Gaussians in the rest frame, centered at different $p_0 = (\pm p_{x0}, 0, p_{z0})$ in the momentum space. Boost angles $\theta_a < 90^\circ$, $\theta_b = 90^\circ$ and $\theta_a > 90^\circ$ correspond to rest frame momenta $p_0$ and are are shown for one peak of each state.
Particle in different boost scenarios

Figure 9. (a) Spin entropy for $x$-symmetric Gaussians with $\sigma/m = 1$. Two boost geometries $\theta_e, \nu_1 = 0.999$ and $\theta_f, \nu_1 = 0.99995$, with $\theta > 90^\circ$ are shown. (b) Schematic representation of Gaussians in the rest frame, centered at different $p_0 = (\pm p_{x0}, 0, p_{z0})$ in the momentum space. Boost angles $\theta_a < 90^\circ$, $\theta_b = 90^\circ$ and $\theta_a > 90^\circ$ correspond to rest frame momenta $p_0$ and are shown for one peak of each state.
From a geometric point of view

- Vectors in the Hilbert space, $|p\rangle|\lambda\rangle \in L(\mathbb{R}^3) \otimes \mathbb{C}^2$, as vector fields $\lambda(p)$ on the mass-shell of a particle with mass $m$

Figure 10. (a) Constant spin field in the rest frame. (b) Wigner rotated spin field in the boosted frame.
From a geometric point of view

- Spin state $\rho_S$, found by tracing out momentum, as a (possibly infinite) convex sum of spin projection operators $|\lambda(p)\rangle\langle\lambda(p)| = \Pi_\lambda(p)$ over the support of the Gaussian

$$\rho_S = \alpha(-p_2)\Pi_\lambda(-p_2) + \alpha(-p_1)\Pi_\lambda(-p_1) + \alpha(p_1)\Pi_\lambda(p_1) + \alpha(p_2)\Pi_\lambda(p_2)$$

(9)

where the coefficients satisfy $\sum_i \alpha(p_i) = 1$

Figure 11. Tracing out momentum amounts to forming a convex sum of spins $\Pi_\lambda(p_i)$ that are Wigner rotated by $\omega_i \equiv \omega(p_i)$, here represented on the Bloch sphere. The resulting spin state $\rho_S$ (boldface arrow) is generally mixed.
Explaining behavior: saturation

- When two boosts at angle $\theta$ approach the speed of light, Wigner rotation asymptotically approaches a particular maximum value $\omega_m$ (see FIG. 5).
- This implies that each individual spin of the field asymptotically approaches a particular $p$-dependent maximum rotation angle $\omega_m(p)$ as both boosts approach the speed of light.
- Since entropy is a monotonic function of spin, its behavior follows the same pattern: it approaches asymptotically a particular level as rapidity grows arbitrarily large.
Explaining behavior: *level* of saturation

- Why does saturation reach *different levels* for Gaussians initially centered at different $p_{z0}$?
- The maximum value of Wigner rotation $\omega_m$ depends on the angle $\theta$ between two boosts. This means boost angle $\theta$ is determined by the center $p_0$ of the Gaussian wave packet.
- However, specifying $\theta$ amounts to setting a bound on the maximum value of rotation, that is, specifying $\omega_m$. The latter, in turn, sets a bound to the maximum rotation of spin operators on the Bloch sphere in FIG. 11 or, equivalently, entropy.
- As a result, for two Gaussians with angles $\theta_a$ and $\theta_b$, where $\theta_a < \theta_b$, entanglement saturates at a lower level for $\theta_a$ than for $\theta_b$. 
Explaining behavior: the bump effect

Why is it that for boost geometries with $\theta \geq 90^\circ$ entanglement initially reaches a maximum value and thereafter saturates at a lower value? Because spins ‘over-rotate’.

Consider the scenario with $v_1 = 0.999, \theta = 161^\circ$ in FIG. 10a. Initially, as rapidity starts to grow, spins start to rotate in opposite directions at either Gaussian and so entanglement starts to increase in line with the explanation above. At $\xi = 2.4$, the effective spin of either Gaussian in FIG. 11 has rotated by $|\omega| = 90^\circ$, hence the spins of the left and right Gaussians become orthogonal and entanglement attains the maximum value 1. Now as rapidity increases further, spins ‘over-rotate’, becoming again non-orthogonal and spin entropy starts to decrease.

Eventually the Wigner rotation attains a maximum value $\omega_m$ and entropy saturates at a value less than 1.

In the limiting case of large boosts $v_1, v_2 \rightarrow 1$, narrow Gaussians, $\sigma \rightarrow 0$ and boost angles $\theta \rightarrow 180^\circ$, the boosted state approaches a product state and entanglement vanishes.
Conclusion

- Entanglement is observer dependent and exhibits rich behavior in the relativistic setting
- Entanglement change can be offered a natural geometric explanation
- Maximal entanglement between spin and momentum components of a single particle can be achieved with sub-luminal boosts
- Boost parameters must be chosen carefully as too large boosts may lead to deterioration of entanglement
- Effect persists for realistic states, i.e. Gaussian wave packets

[Palge, V and Dunningham, J. Generation of maximally entangled states with sub-luminal Lorentz boosts. Physical Review A 85, 042322 (2012)]
Bibliography


