Generation of maximally entangled states with sub-luminal Lorentz boosts

Veiko Palge and Jacob Dunningham

Quantum Information Group
School of Physics and Astronomy
University of Leeds

pyvp@leeds.ac.uk

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Outline

Motivation

- Entanglement as resource in quantum information theory. Enables to create technology that performs information processing tasks which are beyond the limits of the classical realm
- The theory of entanglement given in non-relativistic QM, i.e. Galilean spacetime. But spacetime is ultimately relativistic theory, hence need to study relativistic entanglement

Historical background

- In 1926, Thomas discovers Thomas precession and derives the relativistic correction for the hydrogen atom [5, 6]
- In 1939, Wigner provides a comprehensive treatment of unitary representations for the Lorentz group [7]
- First papers on relativistic quantum information on single and two particle entanglement around 2000 [1, 4, 2, 3]

Entanglement

- The term entanglement (*Verschränkung*) was coined by Schrödinger (1935) in the discussion of the EPR (1935) experiment
- Bell (1964) shows that entangled quantum states display correlations that cannot be reproduced by classical physics
- Entanglement is regarded as the feature that distinguishes quantum and classical physics
- Quantum information theory uses entangled states to perform information processing tasks that are beyond classical realm

Entanglement II

- A quantum state $|\psi\rangle$ is entangled if it cannot be written as a product of its constituents. For bipartite systems

$$|\psi^{AB}\rangle \neq |\psi^{A}\rangle \otimes |\psi^{B}\rangle \equiv |\psi^{A}\rangle|\psi^{B}\rangle \tag{1}$$

- Example: for a two-level bipartite system, $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$
  - $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $\frac{1}{\sqrt{2}}|0\rangle\left(|0\rangle + |1\rangle\right)$ etc are product
  - $\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle \pm |1\rangle|1\rangle\right)$, $\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle \pm |1\rangle|0\rangle\right)$ etc are entangled
- In general, characterization of entanglement is a non-trivial task. Bipartite entanglement is now well understood. The theory of multi-partite entanglement an active field of research

Galilean spacetime

- Galilei boosts form a group: for any two (non-collinear) Galilei boosts $G(v_1), G(v_2)$, there exists a third $G(v_3)$ boost such that

$$G(v_2)G(v_1) = G(v_3)\ , \quad v_3 = v_1 + v_2\ . \tag{2}$$

Adding velocities non-relativistically, $v_1 + v_2 = v_3$

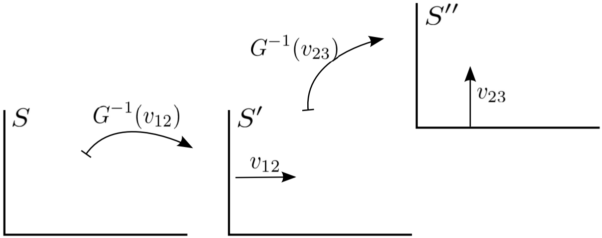


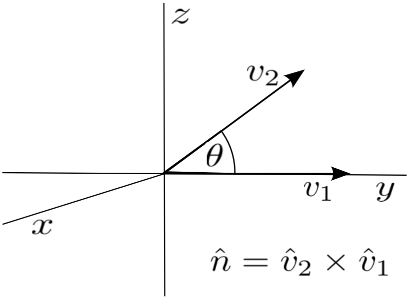
Figure 1. Three Galilei frames.

Minkowski spacetime

- Lorentz boosts *do not* form a group: two non-collinear Lorentz boosts

$$\Lambda(v_2)\Lambda(v_1) \neq \Lambda(v_3)\ , \quad \Lambda(v_2)\Lambda(v_1) = R(\omega)\Lambda(v_3) \tag{3}$$

where $R(\omega) \in SO(3)$ is Wigner's rotation for massive particles



Rotation angle

$\tan\frac{\omega}{2} = \frac{\sin\theta}{\cos\theta + D(v_1, v_2)}$

Kinematic effect due to spacetime structure, depends solely on the velocity and acceleration of the object

Non-relativistic spacetime

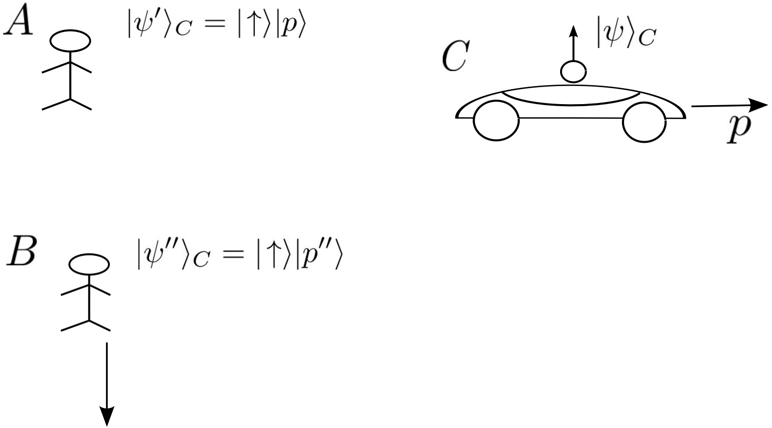


Figure 2. Non-relativistic frames.

Relativistic spacetime

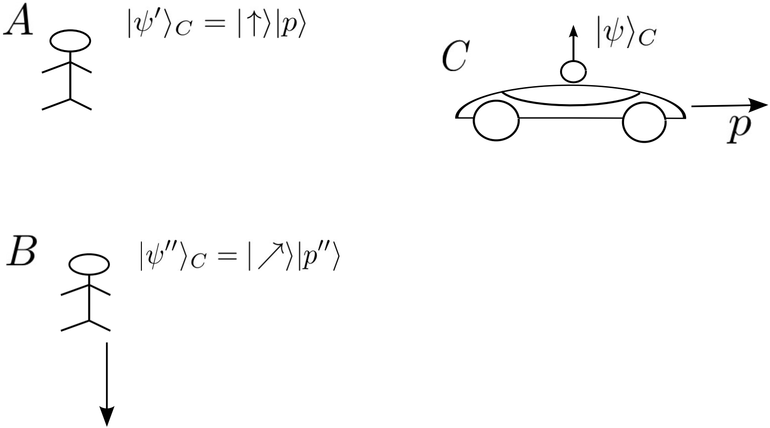


Figure 3. Relativistic frames.

Wigner rotation: example 1

- Bob and Alice to meet. Alice gives Bob instructions: $B_y(v_1)B_x(v_0)$. But Bob has a lot on his mind and takes instead the route $B_x(v_0)B_y(v_1)$. Alice and Bob never meet. . .

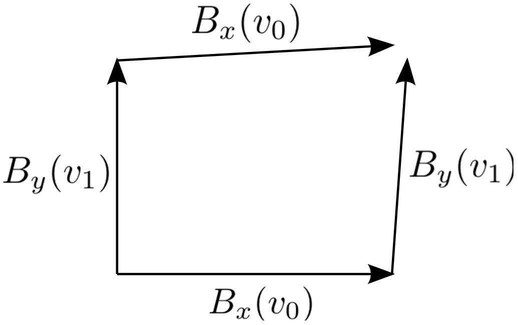


Figure 4. Two Lorentz boosts non-commutative.

Wigner rotation properties

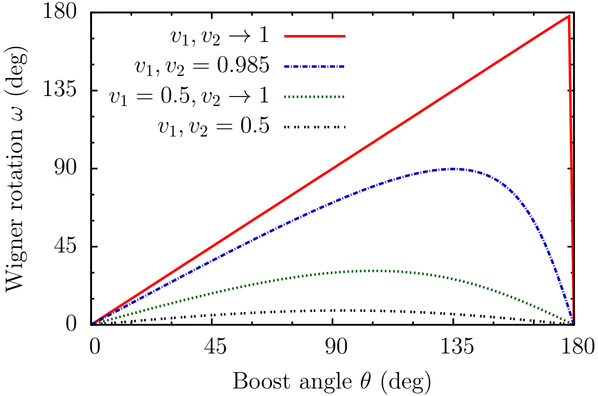


Figure 5. Dependence of Wigner rotation on the angle θ between two boosts.

Lorentz boosted single spin-1/2 particle

- Focus on a single massive spin-1/2 particle with momentum
  "Assuming that spin and momentum are initially in a product state, will they become entangled after two non-collinear Lorentz boosts?"
  Generic state

$$|\psi\rangle = \sum_{\lambda} \int \psi_{\lambda}(p) |p\rangle |\lambda\rangle \, \mathrm{d}\mu(p) \tag{4}$$

  To Lorentz boosted observer O'' the state of the particle appears transformed

$$\psi_{\lambda}(p) \mapsto \psi''_{\lambda}(p) = \sum_{\kappa} U_{\lambda\kappa}(R(\Lambda, \Lambda^{-1}p)) \psi_{\kappa}(\Lambda^{-1}p) \tag{5}$$

- Example

$$(|p\rangle + |-p\rangle) \, |0\rangle \xrightarrow{\Lambda} |p''\rangle |\nwarrow\rangle + |-p''\rangle |\nearrow\rangle \tag{6}$$

Lorentz boosted single spin-1/2 particle

Since we are interested in spin, trace out the momentum

$$\begin{aligned} \rho_S'' &= \mathrm{Tr}_p \left(U(\Lambda)|\psi\rangle\langle\psi|U^\dagger(\Lambda) \right) \\ &= \sum_{\lambda\kappa} \int \psi_\lambda''(p)\psi_\kappa''^*(p)|\lambda\rangle\langle\kappa|\, \mathrm{d}\mu(p) \end{aligned} \tag{7}$$

To quantify entanglement, we calculate the von Neumann entropy of spin

$$S(\rho_S'') = -\mathrm{Tr}(\rho_S'' \log \rho_S'') \tag{8}$$

Lorentz boosted single spin-1/2 particle

- Rest frame state: wavefunction x-symmetric Gaussian centred at $p_0 = (\pm p_{x0}, 0, p_{z0})$, spin z-up and boost in the z-direction

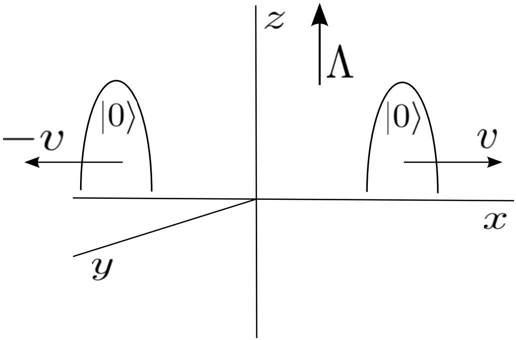


Figure 6. Single spin-1/2 particle with spin and momentum. Lorentz boost in the z-direction.

Particle in different boost scenarios

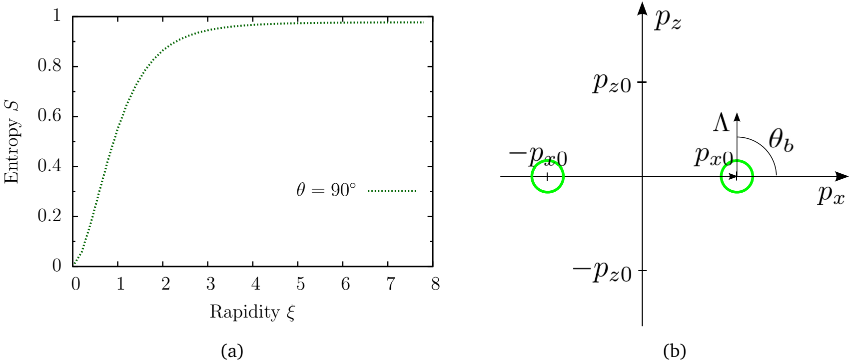


Figure 7. (a) Spin entropy for an x-symmetric Gaussian with $\sigma/m = 1$ with boost geometry $\theta = 90°$ and $v_1 = 0.985$. (b) Schematic representation of Gaussian in the rest frame.

Particle in different boost scenarios

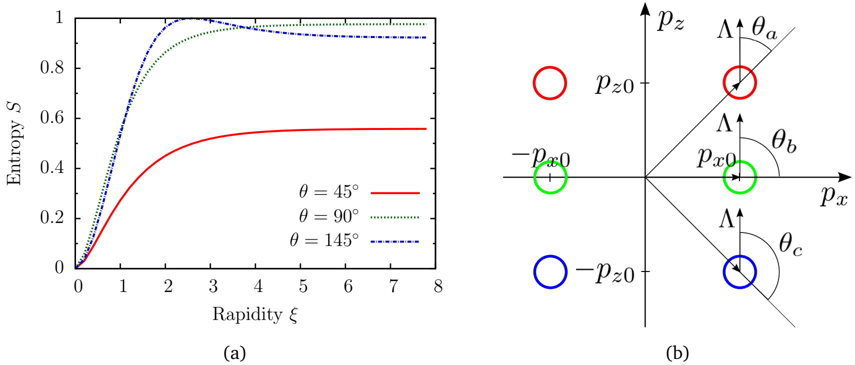
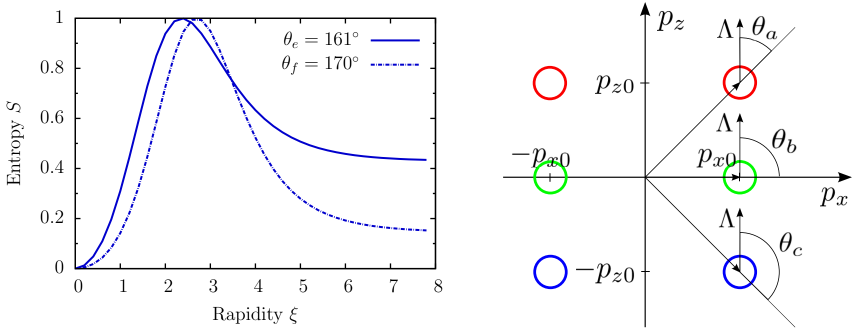


Figure 8. (a) Spin entropy for x-symmetric Gaussians with $\sigma/m = 1$. Three boost geometries with different θ_i are shown, all $v_1 = 0.985$. (b) Schematic representation of Gaussians in the rest frame, centered at different $p_0 = (\pm p_{x0}, 0, p_{z0})$ in the momentum space. Boost angles $\theta_a < 90°$, $\theta_b = 90°$ and $\theta_a > 90°$ correspond to rest frame momenta p_0 and are are shown for one peak of each state.

Particle in different boost scenarios



(a) (b)

Figure 9. (a) Spin entropy for x-symmetric Gaussians with $\sigma/m = 1$. Two boost geometries $\theta_e, v_1 = 0.999$ and $\theta_f, v_1 = 0.99995$, with $\theta > 90°$ are shown. (b) Schematic representation of Gaussians in the rest frame, centered at different $p_0 = (\pm p_{x0}, 0, p_{z0})$ in the momentum space. Boost angles $\theta_a < 90°$, $\theta_b = 90°$ and $\theta_a > 90°$ correspond to rest frame momenta p_0 and are are shown for one peak of each state.

From a geometric point of view

- Vectors in the Hilbert space, $|p\rangle|\lambda\rangle \in L(\mathbb{R}^3) \otimes \mathbb{C}^2$, as vector fields $\lambda(p)$ on the mass-shell of a particle with mass m

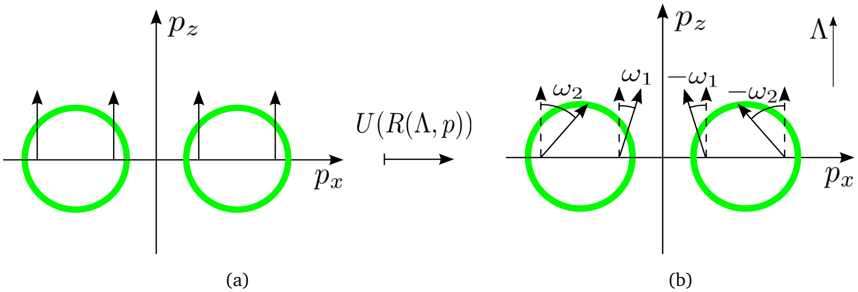


Figure 10. (a) Constant spin field in the rest frame. (b) Wigner rotated spin field in the boosted frame.

From a geometric point of view

- Spin state ρ_S, found by tracing out momentum, as a (possibly infinite) convex sum of spin projection operators $|\lambda(p)\rangle\langle\lambda(p)| = \Pi_\lambda(p)$ over the support of the Gaussian

$$\begin{aligned}\rho_S = \alpha(-p_2)\Pi_\lambda(-p_2) + \alpha(-p_1)\Pi_\lambda(-p_1) \\ + \alpha(p_1)\Pi_\lambda(p_1) + \alpha(p_2)\Pi_\lambda(p_2)\end{aligned} \tag{9}$$

where the coefficients satisfy $\sum_i \alpha(p_i) = 1$

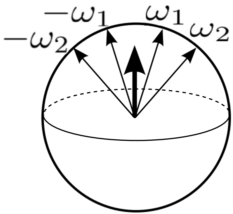


Figure 11. Tracing out momentum amounts to forming a convex sum of spins $\Pi_\lambda(p_i)$ that are Wigner rotated by $\omega_i \equiv \omega(p_i)$, here represented on the Bloch sphere. The resulting spin state ρ_S (boldface arrow) is generally mixed.

Explaining behavior: saturation

- When two boosts at angle θ approach the speed of light, Wigner rotation asymptotically approaches a particular maximum value ω_m (see FIG. 5).
- This implies that each individual spin of the field asymptotically approaches a particular p-dependent maximum rotation angle $\omega_m(p)$ as both boosts approach the speed of light.
- Since entropy is a monotonic function of spin, its behavior follows the same pattern: it approaches asymptotically a particular level as rapidity grows arbitrarily large.

Explaining behavior: *level* of saturation

- Why does saturation reach *different levels* for Gaussians initially centered at different p_{z0}?
- The maximum value of Wigner rotation ω_m depends on the angle θ between two boosts. This means boost angle θ is determined by the center p_0 of the Gaussian wave packet.
- However, specifying θ amounts to setting a bound on the maximum value of rotation, that is, specifying ω_m. The latter, in turn, sets a bound to the maximum rotation of spin operators on the Bloch sphere in FIG. 11 or, equivalently, entropy.
- As a result, for two Gaussians with angles θ_a and θ_b, where $\theta_a < \theta_b$, entanglement saturates at a lower level for θ_a than for θ_b.

Explaining behavior: the bump effect

- Why is it that for boost geometries with $\theta \geq 90^\circ$ entanglement initially reaches a maximum value and thereafter saturates at a lower value? Because spins 'over-rotate'.
- Consider the scenario with $v_1 = 0.999$, $\theta = 161^\circ$ in FIG. 10a. Initially, as rapidity starts to grow, spins start to rotate in opposite directions at either Gaussian and so entanglement starts to increase in line with the explanation above. At $\xi = 2.4$, the effective spin of either Gaussian in FIG. 11 has rotated by $|\omega| = 90^\circ$, hence the spins of the left and right Gaussians become orthogonal and entanglement attains the maximum value 1. Now as rapidity increases further, spins 'over-rotate', becoming again non-orthogonal and spin entropy starts to decrease.
- Eventually the Wigner rotation attains a maximum value ω_m and entropy saturates at a value less than 1.
- In the limiting case of large boosts $v_1, v_2 \to 1$, narrow Gaussians, $\sigma \to 0$ and boost angles $\theta \to 180^\circ$, the boosted state approaches a product state and entanglement vanishes.

Conclusion

- Entanglement is observer dependent and exhibits rich behavior in the relativistic setting
- Entanglement change can be offered a natural geometric explanation
- Maximal entanglement between spin and momentum components of a single particle can be achieved with sub-luminal boosts
- Boost parameters must be chosen carefully as too large boosts may lead to deterioration of entanglement
- Effect persists for realistic states, i. e. Gaussian wave packets

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