

EXTENSIONS:

From topological to noncommutative

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a physical system



A THEORY OF PHYSICS



a mathematical model

- i.e. pendulum, atom, galaxy, human, etc.
- theory specifies regime of applicability

- CM, QM, GR, etc

- mathematical structures (manifolds, vector spaces...)
- and physical interpretations
- used to generate predictions

Quantum gravity will have a regime of applicability which includes both quantum and general relativistic phenomena. It will require genuinely new mathematical structures to express the conceptual content.

Every mathematical model will, either explicitly or implicitly, contain a description of:

1) all the (distinguishable) possible ways a system can exist, i.e. states

2) what quantities can be measured, i.e. observables

The collection of observables ($C^\infty(M, \mathbb{R})$ $B(H)$) have algebraic structure... they can be added/multiplied to get new quantities.

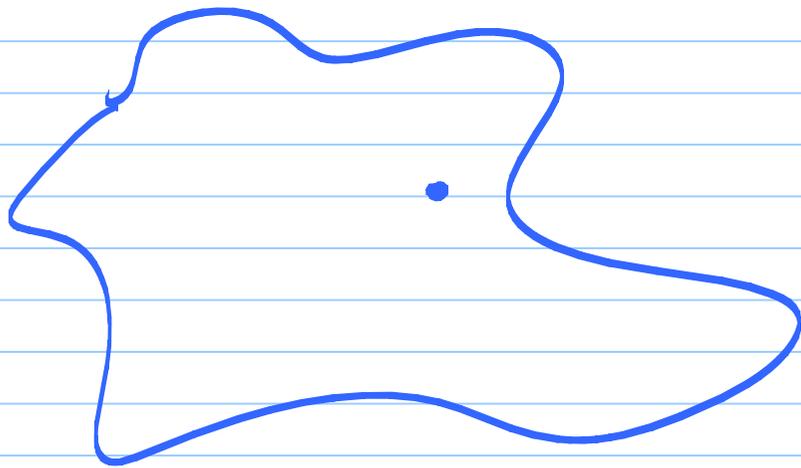
The collection of states (M, H) has geometric structure, i.e. two states are "close" when they have similar properties.

States and observables are in duality.

an observable provides a valuation for each state

the value of observable O for a state s .

Classical physics can be expressed geometrically.



\mathbb{R}

observables = real-valued
functions on
the space
(continuous, at least)

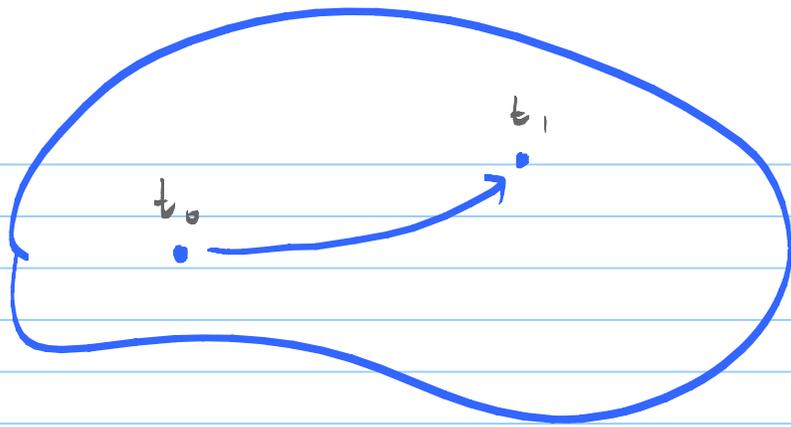
is a state space
"geometry / shape"

i.e. set of points +
extra structure

points = physical pure
states of the system



specifies experimental outcomes
for all observables with
certain precision.



time evolution given by a path in the shape

i.e. in Hamiltonian / symplectic formulations, time evolution is given by flows generated by the energy / Hamiltonian function and the symplectic form

the physics of the system is encoded by the geometry

Quantum mechanics cannot admit such an elegant description. (caveats: non-contextuality, etc)

This, essentially, is the content of the Kochen-Specker Theorem.

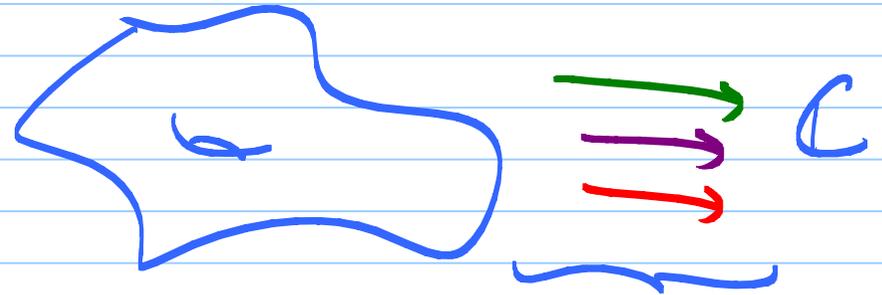
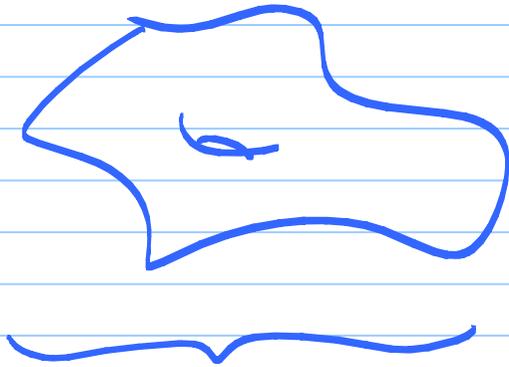
But what if we allow ourselves some liberty in what we consider to be a "geometric space"

Abridged Historical evolution: ^{axioms} Greeks \rightsquigarrow ^{analysis} Descartes \rightsquigarrow ^{curved} Riemann

Poincaré · Hausdorff · Hilbert · Grothendieck ...
algebraic top. sp. ∞ -dim Moving away from "set of points"

VAGUE PROPOSAL: a space is an object which is studied using "geometric tools/intuition"

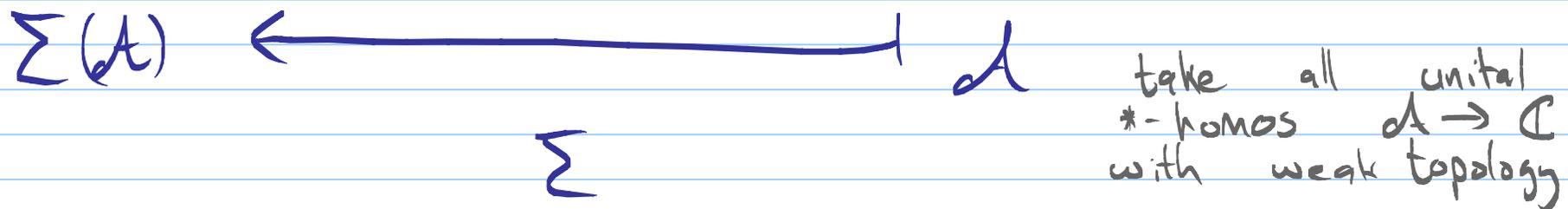
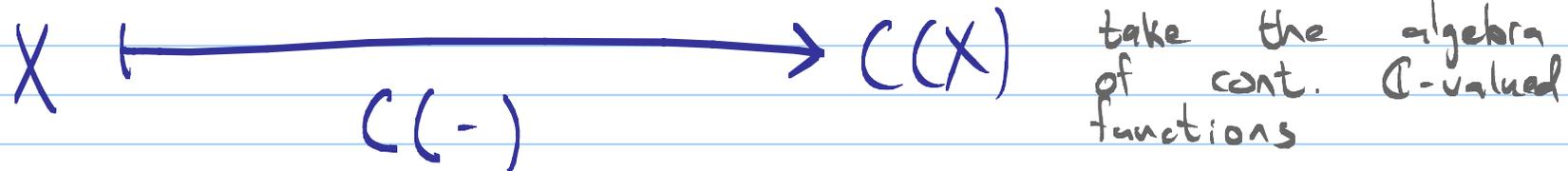
GEL'FAND DUALITY



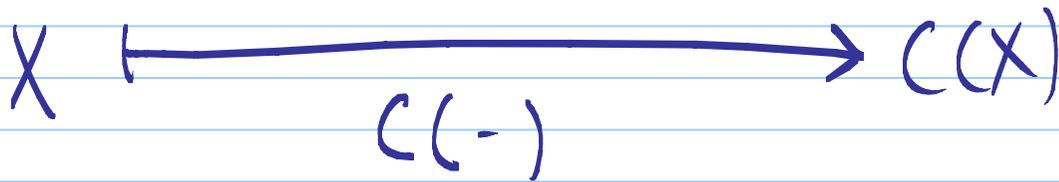
C. H. topological spaces
+ cont. f'ns

commutative, unital C^* -algebras
+ unital $*$ -homomorphisms

an equivalence of categories



Physically...



state space \rightarrow
algebra of observables
(which are the self-adjoint elements)



algebra of observables
 \rightarrow underlying pure state space

In QM, our algebra of observables is non-commutative

The "geometry" of a noncommutative algebra of observables might be called, by analogy, the "state space" of the quantum system

Noncommutative (operator) geometry - NCG - is already an active field of deep mathematical research.

It is actually, in practice, algebra! Operator algebras.

The essential idea: noncommutative C^* -algebras can be thought of as an exotic sort of topological space using Gel'fand duality as the guiding analogy.

i.e. an algebra \mathcal{A} is the algebra of continuous functions from the "noncommutative space of \mathcal{A} " to \mathbb{C} .

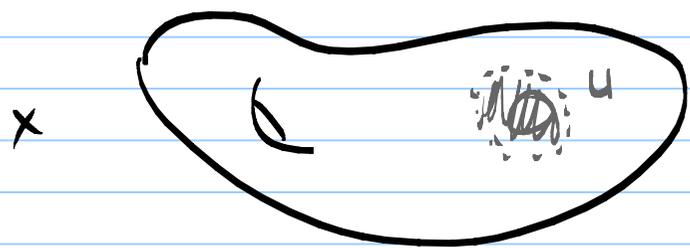
There is no actual topological space X s.t.

$\mathcal{A} = C(X)$ when \mathcal{A} is commutative. We have a

"phantom space" and must work with the algebra of f's.

How do we do this? Use Gel'fand duality
to rewrite geometric concepts of functions and tools
in terms of the algebra of functions.

A simple example: open sets.

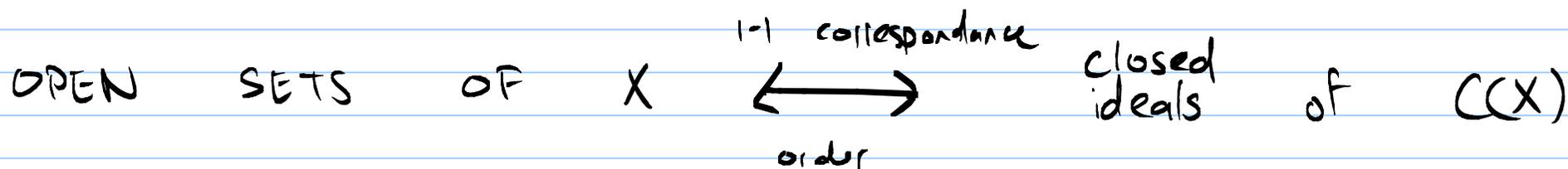


Consider all functions which
vanish outside U .

This set is obviously closed under $+$.

It is also easily seen to absorb $C(X)$ - it is an
ideal of $C(X)$.

Conversely, ideals of $C(X)$ determine open sets of X .



NON COMMUTATIVE DICTIONARY

Topology

continuous \mathbb{C} -valued function

continuous \mathbb{R} -valued function

open set

vector bundle

homeomorphism

disjoint union

cartesian product

Borel measure

integral

1-point compactification

Stone - Čech compactification

Algebra

element of the algebra

self-adjoint element

closed ideal

finite, projective module

isomorphism

direct sum

minimal tensor product

positive functional

trace

unitization (max)

multiplier algebra

and so on...

A geometrical dual to noncommutative C^* -algebras would provide, for QM, the direct analogue of classical state space.

It would also provide a picture of spacetime as posited in models of quantum gravity based on replacing spacetime manifolds with non-commutative geometric manifolds.

We investigate a proposal for quantum state space postulated in the "topos formalism" of QM (Isham, Butterfield, Döring) by studying its suitability as a geometric dual for noncommutative C^* -algebras.

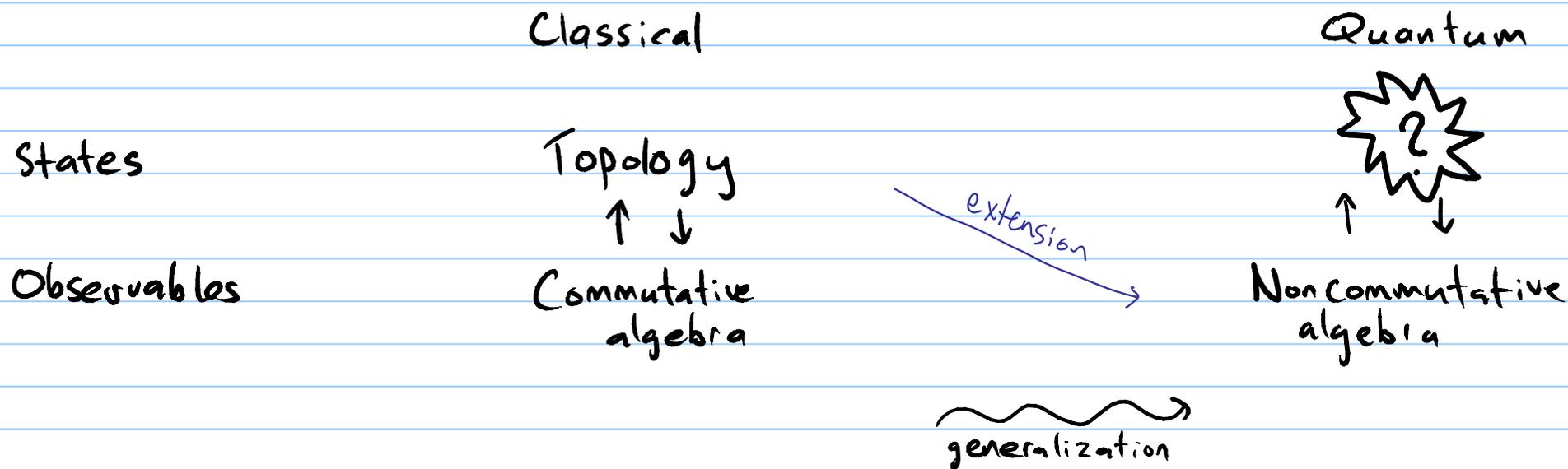
Given a noncommutative C^* -algebra A , one constructs a presheaf whose objects are the spectra of the commutative subalgebras of A , i.e. the state spaces of "classical subsystems".

$A \rightarrow$ diagram of commutative subalgebras ordered by inclusions \rightarrow spectra of this diagram

Associates to A a diagram of topological spaces

This diagram leads to a natural way of extending classical/topological concepts to quantum/noncommutative concepts via universal properties.

Serve as a classical/quantum or geometry/Non-comm dictionary?



$$F : \text{Top} \rightarrow \mathbb{C} \quad \xrightarrow{\text{extension}} \quad \tilde{F} : C^*\text{-alg} \rightarrow \mathbb{C}$$

$$\tilde{F}|_{\text{Com } C^*} \cong F \circ \Sigma$$

$$\text{Id} : \text{Top} \rightarrow \text{Top}$$

i.e. state space

$$\tilde{\text{Id}} : \nu\mathbb{N} \rightarrow \text{Top}$$

$$= \begin{cases} \Sigma \\ \emptyset \end{cases}$$

commutative

non commutative (M_2)

This is equivalent to the Kochen - Specker Theorem.

$D: \text{Top} \rightarrow \text{Convex Sets}$

assigns to a space: the
convex set of all probability
distributions (gas)

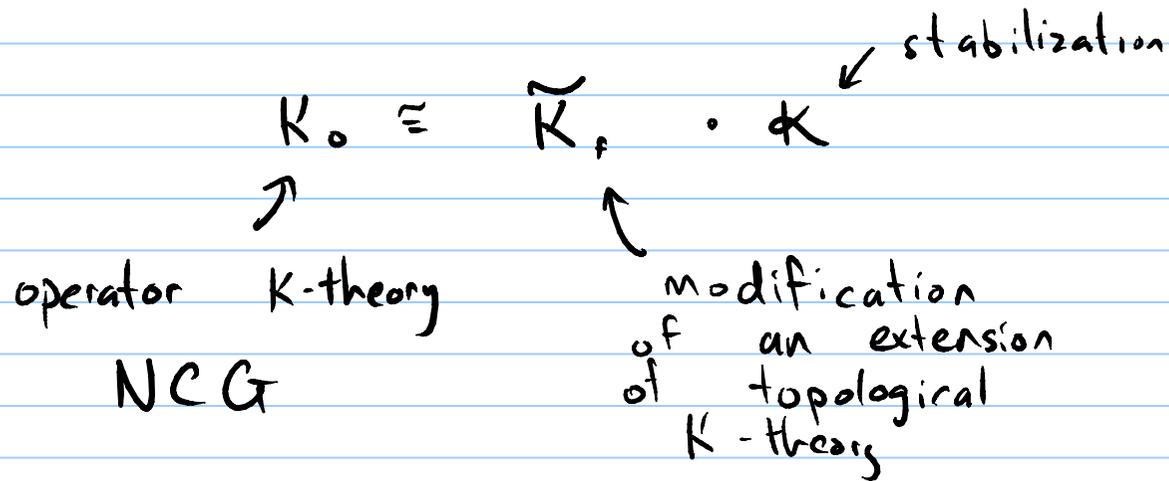
$\tilde{D}: vN \rightarrow \text{Convex Sets}$

assigns to an algebra what is more
commonly known as state space (positive, linear
functionals of norm 1), i.e. mixed states.

This is equivalent to Gleason's Theorem
and gives an interesting picture of quantum states
as consistent families of classical distributions.

$$K: \text{Top} \rightarrow \text{Ab}$$

Topological K-theory
Cohomology theory



Suggests a close connection between extension process and canonical method of generalizing geometric tools in NCG.

THANKS!

COMPLETENESS OF QM



CAUSALITY
and
REALITY

are incompatible

MEASURE
STRUCTURE ON
 Σ
ON TIC = DELTA

LOCALITY \Rightarrow OBS.
PROP. ON TIC