The ZX-calculus is complete for stabilizer quantum mechanics

Miriam Backens

Department of Computer Science
University of Oxford

Quantum Fields, Gravity & Information 2013

Outline

Background

- Stabilizer quantum mechanics
- The ZX-calculus

The completeness proof

- “Normal forms” for ZX-calculus diagrams
- Comparing stabilizer ZX-calculus diagrams

Outlook

Example

Operations in stabilizer quantum mechanics

- Preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Measurements in computational basis

Graph states

Definition

Given a finite simple undirected graph G with set of vertices V and set of edges E, the corresponding graph state $|G\rangle$ is the state created as follows:

- for each vertex v in V, prepare a qubit in the state $|+\rangle = H|0\rangle$, then
- for each edge $\{u, v\}$ in E, apply a controlled-Z operation to the qubits u and v.

E.g.

- $\bullet$ corresponds to $|+\rangle$
- $\bullet\!\!-\!\!\!-\!\!\bullet$ corresponds to

$$\Lambda Z(|+\rangle \otimes |+\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

Local Clifford equivalences

The local Clifford group consists of all tensor products of elements of the single-qubit Clifford group $\langle S, H \rangle$.

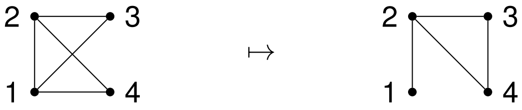
Theorem (Van den Nest et al., 2004)

Any stabilizer state is local Clifford-equivalent to some graph state.

Theorem (Van den Nest et al., 2004)

Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.

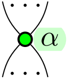
A local complementation about a vertex v inverts the subgraph generated by the neighbourhood of v: e.g.



under a local complementation about vertex 2

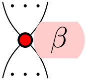
Elements of the stabilizer ZX-calculus

- green spiders with n inputs and m outputs, $\alpha \in \{0, \pi/2, \pi, -\pi/2\}$



$$= \begin{cases} |0\rangle^{\otimes n} \mapsto |0\rangle^{\otimes m} \\ |1\rangle^{\otimes n} \mapsto e^{i\alpha} |1\rangle^{\otimes m} \end{cases}$$

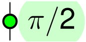
- red spiders with n inputs and m outputs, $\beta \in \{0, \pi/2, \pi, -\pi/2\}$

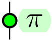


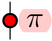
$$= \begin{cases} |+\rangle^{\otimes n} \mapsto |+\rangle^{\otimes m} \\ |-\rangle^{\otimes n} \mapsto e^{i\alpha} |-\rangle^{\otimes m} \end{cases}$$

- Hadamard nodes with one input and one output

$$= \begin{cases} |0\rangle \mapsto |+\rangle \\ |1\rangle \mapsto |-\rangle \end{cases}$$

Note that 

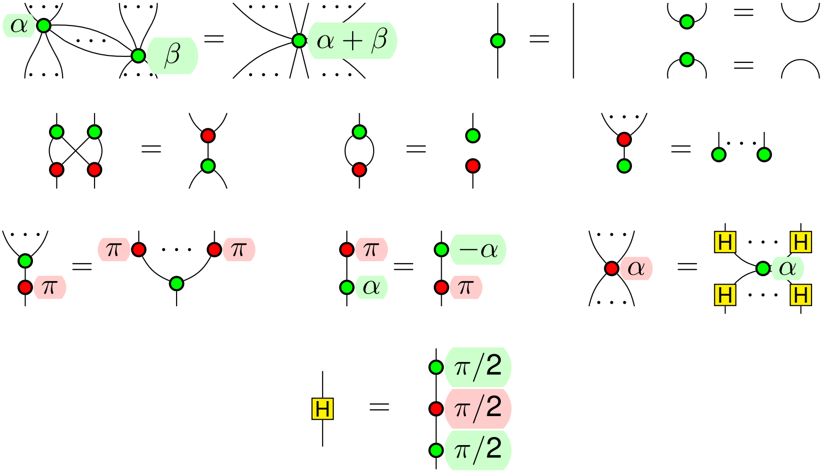
$= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, 

$= Z$ and 

$= X$.

Rules of the ZX-calculus

- ignore non-zero scalar factors (π $= 0 =$ π)
- only the topology matters



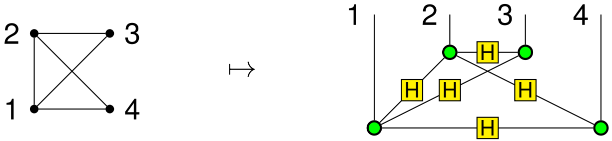
Graph states in the ZX-calculus

Definition

The ZX-calculus diagram of the graph state $|G\rangle$ consists of:

- for each node in G, a green node with one output, and
- for each edge in G, an edge with a Hadamard node on it.

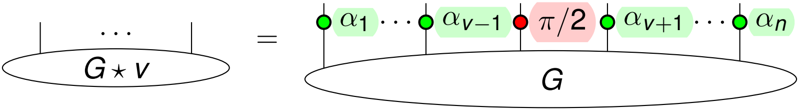
E.g.



Local complementations in the ZX-calculus

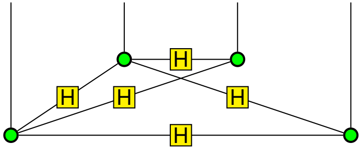
Theorem (Duncan & Perdrix, 2009)

Denote the result of a local complementation about the vertex v in the graph G by $G \star v$. The graph state diagrams for G and $G \star v$ satisfy



where $\alpha_u = -\pi/2$ if $\{u, v\}$ is an edge, $\alpha_u = 0$ otherwise.

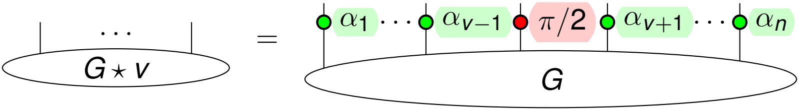
E.g. a local complementation about the 3rd qubit of the previous graph state:



Local complementations in the ZX-calculus

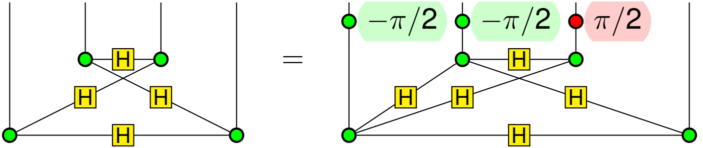
Theorem (Duncan & Perdrix, 2009)

Denote the result of a local complementation about the vertex v in the graph G by $G \star v$. The graph state diagrams for G and $G \star v$ satisfy



where $\alpha_u = -\pi/2$ if $\{u, v\}$ is an edge, $\alpha_u = 0$ otherwise.

E.g. a local complementation about the 3rd qubit of the previous graph state:

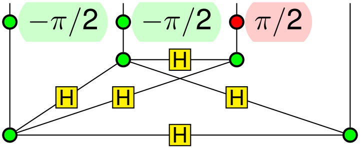


GS-LC diagrams

Definition

A diagram in the stabilizer ZX-calculus is called a *GS-LC diagram* if it consists of a graph state diagram with arbitrary single-qubit Clifford operators (i.e. combinations of phase shifts and Hadamards) applied to each output.

E.g.



Note: GS-LC diagrams don't need to be connected and there may be an arbitrary combination of phase shifts and Hadamard nodes on each output.

Every state diagram is equal to some GS-LC diagram

Theorem

Each ZX*-calculus diagram with no inputs and at least one output can be rewritten to some GS-LC diagram.*

Proof.

- Decompose the diagram into basic spiders

  and single-qubit Clifford operators.
- Diagrams with no inputs must contain at least one copy of , this is a GS-LC diagram.
- For each basic element, applying it to a GS-LC diagram yields a diagram that can be rewritten into GS-LC form.
- Thus, by induction, the theorem holds. □

Completeness for stabilizer state diagrams

The result on the previous slide is the ZX-calculus equivalent of the theorem that any stabilizer state is local Clifford-equivalent to some graph state. Recall from standard stabilizer quantum mechanics:

Theorem (Van den Nest et al., 2004)

Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.

In ZX-calculus terms, this means that two GS-LC diagrams are equal if and only if they are related by a sequence of local complementations. Thus:

Theorem

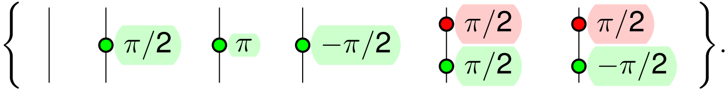
The ZX*-calculus is complete for stabilizer state diagrams.*

Reduced GS-LC diagrams and comparison algorithm

Definition

A diagram in the ZX-calculus is called a *reduced GS-LC diagram* if it is a GS-LC diagram and satisfies the following two conditions:

1. All local Clifford operators belong to the set



2. Two adjacent vertices must not both have local Clifford operators that include red nodes.

Theorem (inspired by Elliott et al., 2008)

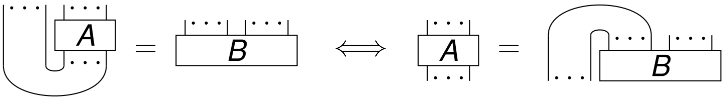
Each ZX-calculus state diagram is equal to some reduced GS-LC diagram. Furthermore, there exists a terminating algorithm that, given a pair of reduced GS-LC diagrams on the same number of qubits, leads to a pair of identical diagrams if and only if the two diagrams represent the same state.

The Choi-Jamiołkowski isomorphism

We've only talked about state diagrams so far. What if a diagram has some non-zero number of inputs?

Theorem (Choi-Jamiołkowski isomorphism)

For any operator A from n to m qubits and for any $n + m$-qubit state B,



The stabilizer ZX-calculus is complete

Theorem

The ZX-calculus is complete for all stabilizer diagrams.

Proof.

Given two ZX-calculus diagrams with n inputs and m outputs each:

- Apply the Choi-Jamiołkowski isomorphism to get two diagrams with $n + m$ outputs each.
- Bring the diagrams into reduced GS-LC form.
- Apply the comparison algorithm for reduced GS-LC diagrams.
- If this yields a pair of identical diagrams, use the Choi-Jamiołkowski isomorphism to transform the sequence of equal state diagrams back into operators.

This yields a sequence of rewrites which transforms one of the original diagrams into the other. □

Outlook

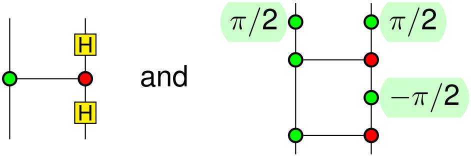
- Extend result to the entire ZX-calculus.
  - Not obvious how to do that or whether it is even possible.
  - Might be able to start with smaller steps, e.g. looking at operators which can be represented exactly in terms of Clifford operators and the T gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

- Implement decision procedure in `Quantomatic`.
- See if result carries over to Spekkens' toy theory.
- There is some redundancy in the laws of the ZX-calculus as given in this talk; what is the minimum number of laws needed to get stabilizer completeness?

Example: the controlled-Z operator

Two ways of writing the controlled-Z operator in quantum circuit notation, translated into the ZX-calculus:



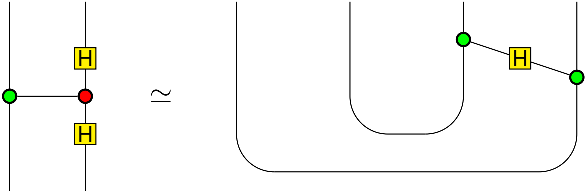
Can we show these two diagrams are the same using the rules of the ZX-calculus?

Example: the controlled-Z operator, first diagram

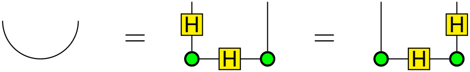
decompose into the basic elements

and single-qubit Clifford operators

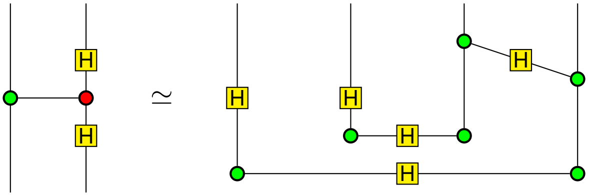
Example: the controlled-Z operator, first diagram



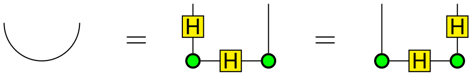
bend the inputs into outputs, noting that



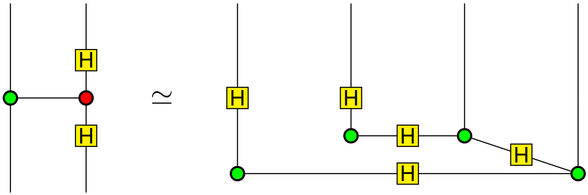
Example: the controlled-Z operator, first diagram



bend the inputs into outputs, noting that

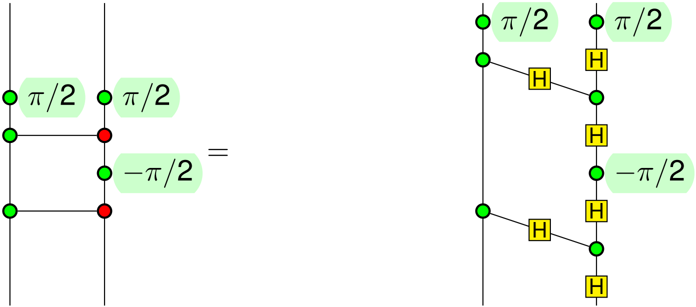


Example: the controlled-Z operator, first diagram



join green nodes

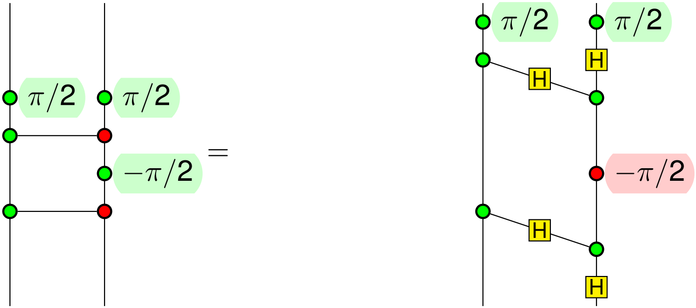
Example: the controlled-Z operator, second diagram



decompose into the basic elements

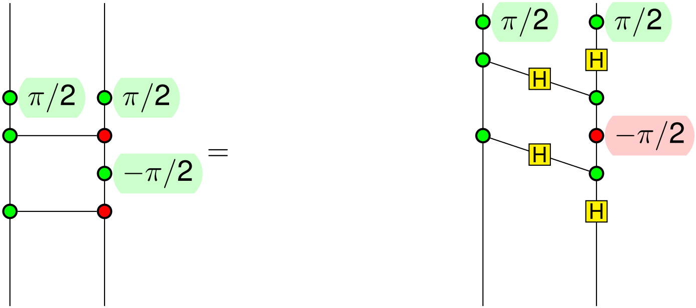
and single-qubit Clifford operators

Example: the controlled-Z operator, second diagram



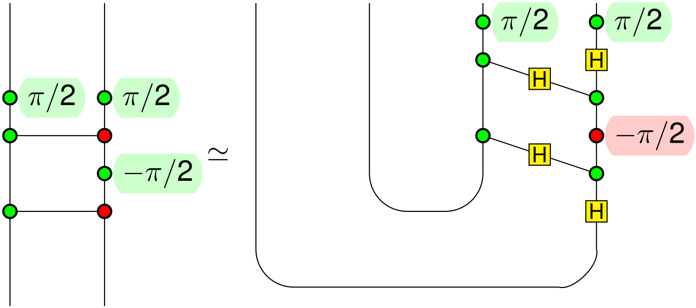
simplify...

Example: the controlled-Z operator, second diagram

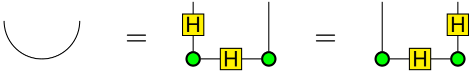


simplify...

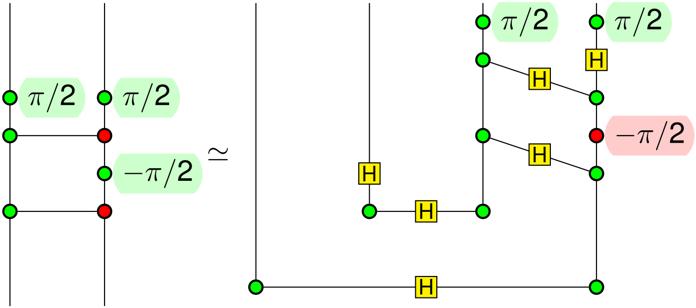
Example: the controlled-Z operator, second diagram



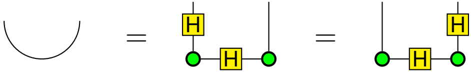
bend the inputs into outputs, noting that



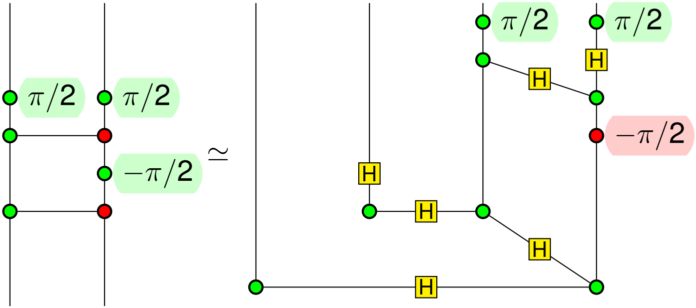
Example: the controlled-Z operator, second diagram



bend the inputs into outputs, noting that

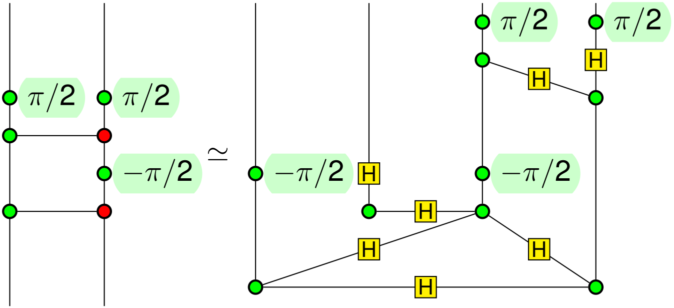


Example: the controlled-Z operator, second diagram



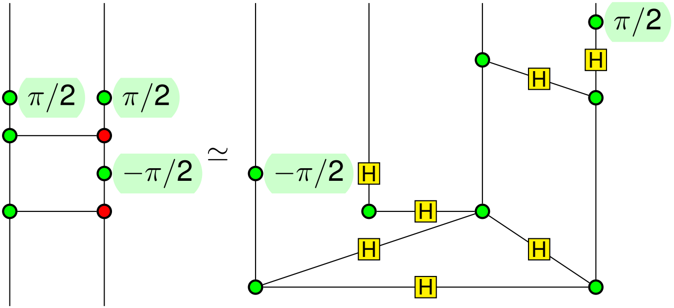
join green nodes

Example: the controlled-Z operator, second diagram



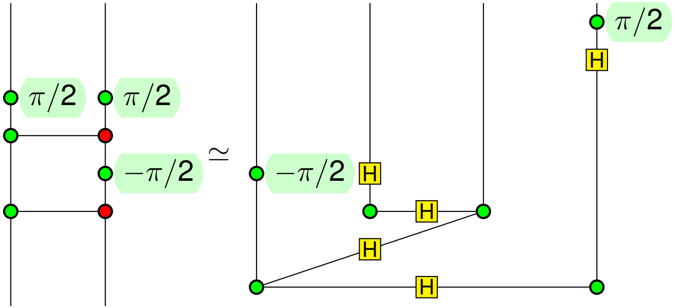
local complementation on 4th qubit

Example: the controlled-Z operator, second diagram



join green nodes

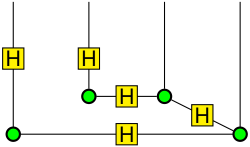
Example: the controlled-Z operator, second diagram



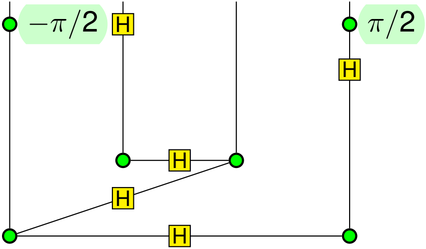
join green nodes

Example: comparison of diagrams

We now have two GS-LC diagrams,

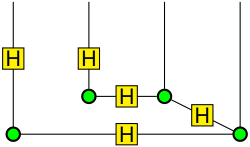


and

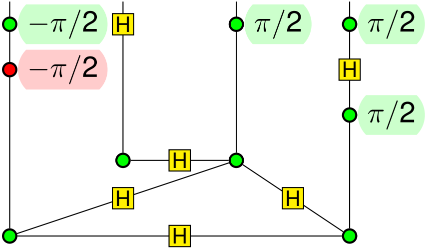


— are these equal?

Example: comparison of diagrams

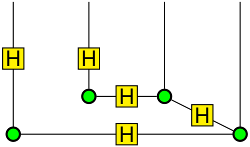


and

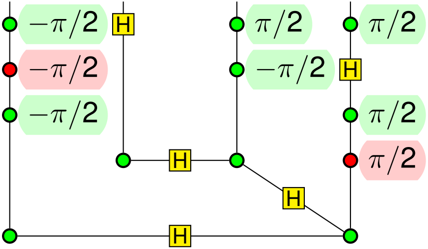


local complementation about the first qubit in the second diagram

Example: comparison of diagrams



and



local complementation about the fourth qubit in the second diagram

$\Longrightarrow$ by spider rule and Euler decomposition rule, the two diagrams are equal