Twisted Correlations In Multipartite Quantum States

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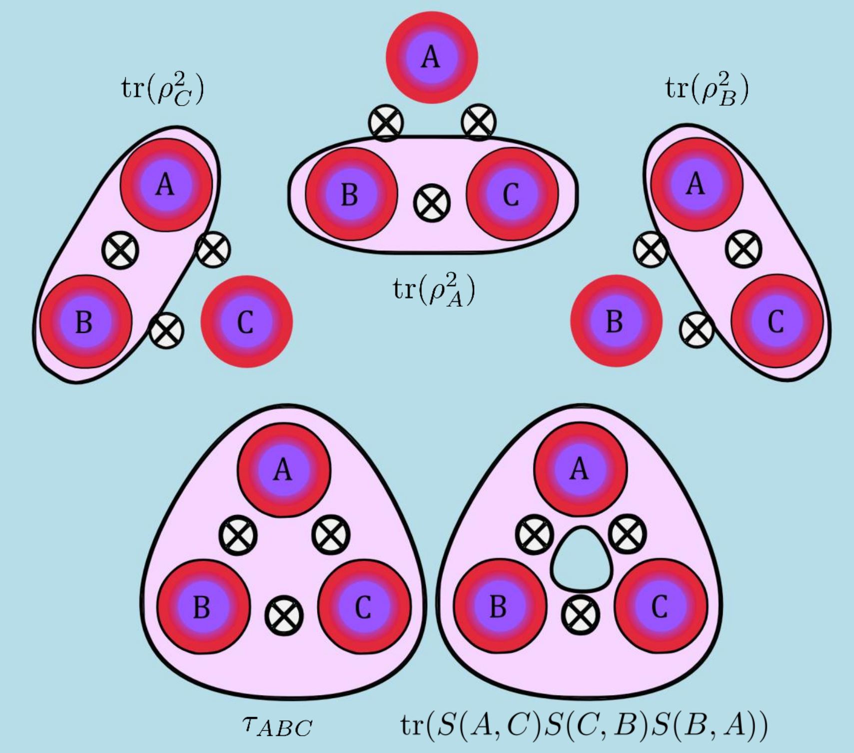
Gauge invariance in quantum information theory

Entanglement Invariants

In quantum information theory, one is principally interested in the nonlocal properties of distributed, multi-partite states $\rho_{ABC...N} \in \mathcal{H}_{\text{local}}^{\otimes N}$ shared between distant observers.

These observers can only affect the joint state through local operations and classical communication[1] (LOCC) between one another.

Since the degrees of freedom in the state are in general more numerous than those available for local operations, there must be some inherently *nonlocal* properties which are invariant in the LOCC paradigm. These are known as *entanglement invariants*[2].



Depiction of the five LOCC invariants for pure three qubit states $|\psi\rangle\langle\psi|_{ABC}$ [2,3]. The shaded area shows how the corresponding entanglement is distributed among subsystems.

Local operations as a gauge group

Transformations which act locally, but leave important global properties unchanged are known as gauge transformations[4]. When considering multi-partite qubit states, the gauge group for LOCC is $SU(2)$. Here, however, we also want to impose invariance under stochastic local operations[1] (SLOCC), leading to the gauge group $SL(2,\mathbb{C})$.

We can further the analogy with gauge field theory by extending the notion of parallel transport to our quantum state. The qubit subsystems play the part of fields on a lattice, and in this way we can generate entanglement invariants using a Wilson loop-like construction[4].

Twisted states

Holonomies and equivalence classes

For an N-qubit state, there are as many invariants generated by the twist as there are independent ordered subsets of 3 or more qubits.

The product of parallel transporters around a loop gives its associated *holonomy*[6]. For a given value of the twist, there is a five parameter equivalence class of holonomies corresponding to the different ways that the twist can arise. Each of these holonomies corresponds to a large set of possible states, reachable through local transformations.

Simple loops - three-qubit states

Tripartite qubit states are characterised by a single twist parameter. It turns out that, whilst product states have trivial twist $\xi = 1$, both classes of entangled pure states (GHZ and W) have $\xi = 0$. That is, one can only symmetrize the bipartite correlations up to a spin flip along some direction. This means that the twist is an entanglement witness for tripartite pure states.

It is possible to generate the full group of holonomies from tripartite states using mixtures of cluster states $|\chi_{ij}\rangle$, symmetric Bell states and distributed singlet states ρ_m :

$$|\chi_{ij}\rangle = \left(|0_j\rangle_A + R_{ij}\sigma_j^B |1_j\rangle_A \right) \otimes \left(|0_j\rangle_B + R_{ij}\sigma_j^C |1_j\rangle_B \right) \otimes\left(|0_j\rangle_C + R_{ij} |1_j\rangle_C \right)/\sqrt{8},$$

$$\rho_m = \tfrac{1}{3}(|\psi^-\rangle\langle\psi^-|_{AB} \otimes \mathbb{I}_C + |\psi^-\rangle\langle\psi^-|_{BC} \otimes \mathbb{I}_A + |\psi^-\rangle\langle\psi^-|_{CA} \otimes \mathbb{I}_B),$$

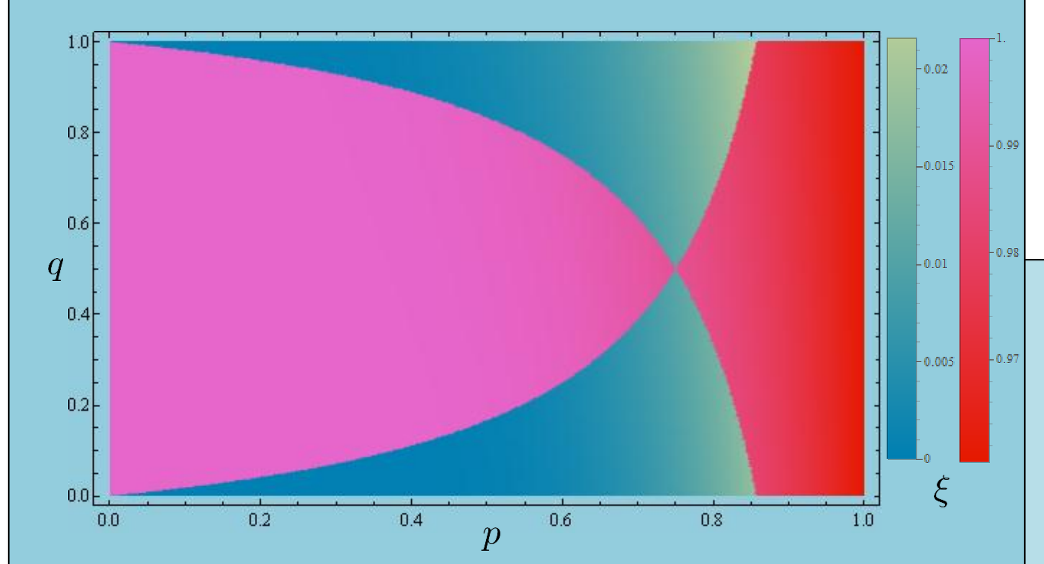
where $(|0_j\rangle + R_{ij}|1_j\rangle)$ is the positive eigenstate of σ_i written in the eigenbasis of σ_j.

A two parameter example

The two parameter ($p,\ q \in [0,1]$) mixture:

$$\rho = \frac{p}{2}(\rho_m + |\chi_{xz}\rangle\langle\chi_{xz}|) + (1-p)\big(q|\psi_y^+\rangle\langle\psi_y^+|_{BC} \otimes \mathbb{I}_A + (1-q)|\psi_y^+\rangle\langle\psi_y^+|_{CA} \otimes \mathbb{I}_B\big),$$

where $|\psi_y^+\rangle = (|0_y 1_y\rangle + |1_y 0_y\rangle)/\sqrt{2}$, has a holonomy composed of a rotation and an orthogonal pure boost ($\prod \Lambda(i,j) \in SO(1,1) \oplus SO(2)$). The two components come from the first and second parts of the mixture respectively.



There are two discontinuities in the twist corresponding to the introduction of sign changes in the singular values of the correlation matrices. This splits the state space into a part with twist similar to truly tripartite GHZ and W states (blue) and a part with twist similar to bipartite entangled and product states (red).

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Parallel transport and the twist

From density to correlation matrices

A two qubit density matrix can be decomposed in terms of the Pauli spin matrices

$$\rho_{AB} = \frac{1}{2}\sum_{i,j=0}^{3} S(A,B)_{ij}\sigma_i^A \otimes \sigma_j^B.$$

$S(A,B)$ is called the correlation matrix, with elements

$$S(A,B)_{ij} = tr(\sigma_i^A \otimes \sigma_j^B \rho_{AB}) = \langle \sigma_i^A \otimes \sigma_j^B \rangle.$$

In the density matrix picture, SLOCC correspond to elements of $SL(2,\mathbb{C})$. In the correlation matrix picture, they become represented by $SO^+(1,3)$, due to the homomorphism $SL(2,\mathbb{C}) \simeq SO^+(1,3)$. This is the gauge group of our theory.
Explicitly, the relation between the group elements is given by

$$\mathcal{V}_{ij} = \frac{1}{2}tr(V^\dagger \sigma_i V \sigma_j).$$

If the density matrix transforms as

$$\rho'_{AB} = (V \otimes W)\rho_{AB}(V^\dagger \otimes W^\dagger)$$

where $V, W \in SL(2,\mathbb{C})$, then $S(A,B)$ evolves as

$$S(a,b)'_{ij} = \mathcal{V}S(a,b)\mathcal{W}^T,$$

where $\mathcal{V},\mathcal{W} \in SO^+(1,3)$.

LSVD and parallel transport

It can be shown[5] that S can always be decomposed as

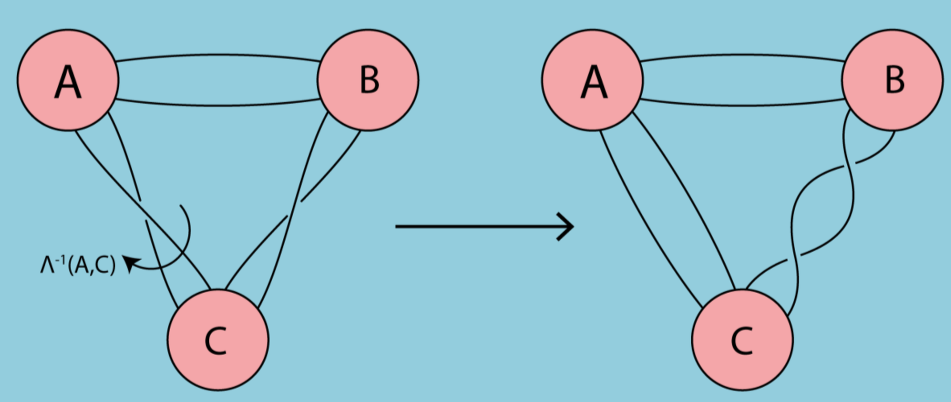
$$S(a,b) = V_a \Sigma_{ab} W_b^T,$$

with $V_a, W_b \in SO^+(1,3)$ corresponding to operations on the qubits A, B, and Σ_{ab} a diagonal matrix, representing a bell diagonal state. This is known as a Lorentz singular value decomposition.

By rewriting the LSVD, we can find a left polar decomposition

$$S(a,b) = (V\eta W^T\eta)(W\Sigma W^T) = \Lambda(a,b)\tilde{S}(a,b).$$

Where $\tilde{S}(a,b)$ is now a symmetrized correlation matrix. $\Lambda \in SO^+(1,3)$ represents an operation applied to the first qubit to symmetrize the two qubit link, and is the parallel transporter in our gauge theory.



The ribbons between each qubit represent the asymmetry of the correlations in the links. Though we can untwist each link individually, by a gauge transformation, we cannot simultaneously symmetrize every link in the loop - the twist is a global property.

Twist - a Wilson loop on the qubit lattice

For an N qubit state, we imagine the qubits as points on an abstract lattice[6], through which we draw a loop. To each link in the loop we can associate a reduced two qubit density matrix, a correlation matrix, and a parallel transporter Λ.

Taking the trace of the product of these transporters around a loop

$$\xi = \frac{1}{4}tr(\Lambda(a,z)...\Lambda(c,b)\Lambda(b,a)),$$

gives us a gauge invariant measure of the global asymmetry in the link correlation matrices, which we call twist. Its definition closely resembles that of the Wilson loop in lattice gauge field theory[4].

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Potential applications to relativistic QM

The twist in curved and accelerated frames

We can calculate the twist on certain states of a quantum field by considering each mode as a subsystem, then defining a loop through them - this technique works as long as we can treat the modes as qubits.

It has been calculated for states of a field confined to an accelerated cavity[7], for which it was found that the vacuum state is untwisted, but higher number states were twisted.

It is also possible to define a related quantity for Gaussian states rather than qubits[8] for which the gauge group is the symplectic group $Sp(2n,F)$.
This definition substantially extends the range of states for which it is possible to define the twist.

Emergent spacetime from qubit networks

If one sums the twist over all plaquettes (smallest loops) in a multipartite qubit state, one arrives at a quantity very close to the Yang-Mills action on a lattice[4]. The fact that the gauge group in this case is (almost) isomorphic to the Lorentz group suggests that one might be able to extract something that looks like space-time from a arbitrarily connected network of qubits.

Though there have been attempts to extract space-time dynamics from spin networks[9], they have not explicitly used the SLOCC gauge invariance which we have outlined here. In doing so, one might find a more direct relation between the geometry of quantum states and that of the universe we live in.

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